Exam Quantum Field Theory
July 10, 2015
Start: 14:00h End: 17:00h

Each sheet with your name and student ID

INSTRUCTIONS: This is a closed-book and closed-notes exam. You are allowed to bring one A4 page written by you on one side, with useful formulas. The exam duration is 3 hours. There is a total of 9 points that you can collect.

NOTE: If you are not asked to **Show your work**, then an answer is sufficient. However, you might always earn more points by answering more extensively (but you can also loose points by adding wrong explanations). If you are asked to **Show your work**, then you should explain your reasoning and the mathematical steps of your derivation in full. Use the official exam paper for *all* your work and ask for more if you need.

USEFUL FORMULAS

The chirality(spin) projectors for spin 1/2 Dirac fermions:

$$P_L = \frac{1-\gamma_5}{2} \quad P_R = \frac{1+\gamma_5}{2} \qquad P_L P_R = 0 \quad P_L^2 = P_L \quad P_R^2 = P_R$$

$$\{\gamma_5, \gamma_\mu\} = 0 \qquad \gamma_5^2 = \mathbb{1}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right) \quad (d=4)$$

$$\mathrm{Tr}(\gamma_5\gamma^\mu) = \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha) = 0$$

Conversion factor: $1\,\mathrm{s}^{-1} = 6.58\times10^{-25}\,\mathrm{GeV}$

1. (2 points) Given the lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3$$

for a real scalar field ϕ with a cubic self-interaction, derive the formula for the complete n-point Green's function

$$G^{(n)}(x_1, \ldots, x_n) = \frac{\int \mathcal{D}\phi\, e^{i\int d^4x\,\mathcal{L}}\,\phi(x_1)\ldots\phi(x_n)}{\int \mathcal{D}\phi\, e^{i\int d^4x\,\mathcal{L}_{g=0}}} \tag{1}$$

using the path integral formulation of the theory. **[Show your work]**

Solution: The path integral in the presence of an external source J is

$$Z[J] = \int \mathcal{D}\phi\, e^{i\int\, d^4x\,\left(\mathcal{L}_0 - \frac{g}{3!}\phi^3 + J\phi\right)} \tag{2}$$

where we indicated with $\mathcal{L}_0$ the lagrangian for the free field. Taylor expand in J to obtain

$$Z[J] = Z[0,0]\sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1\ldots\int d^4x_n\, J(x_1)\ldots J(x_n)\, G^{(n)}(x_1,\ldots,x_n) \tag{3}$$

where the n-point Green's function is given by

$$G^{(n)}(x_1,\ldots,x_n) = \frac{1}{Z[0,0]}\int \mathcal{D}\phi\, e^{i\int d^4x\,\left(\mathcal{L}_0 - \frac{g}{3!}\phi^3\right)}\,\phi(x_1)\ldots\phi(x_n)\,, \tag{4}$$

with

$$Z[0,0] = \int \mathcal{D}\phi\, e^{i\int d^4x\,\mathcal{L}_0} \tag{5}$$

the path integral for $J = 0$ and $g = 0$.

2. (2 points total) Given $G^{(n)}(x_1,\ldots,x_n)$ in (1) for the ϕ^3 theory,

a) (1.5 points) derive the first non zero contribution to the connected four-point function $G_c^{(4)}(x_1,\ldots,x_4)$ in an expansion in powers of the coupling g. Derive the expression in coordinate space only, in terms of the propagator $D(x_i - x_j)$ of the scalar field. **[Show your work]**

Solution:

$$G^{(4)}(x_1,\ldots,x_4) = \frac{1}{Z[0,0]}\int \mathcal{D}\phi\, e^{i\int d^4x\,\left(\mathcal{L}_0 - \frac{g}{3!}\phi^3\right)}\,\phi(x_1)\ldots\phi(x_4) \tag{6}$$

$O(g^0)$ only contains disconnected contributions.

$O(g)$ contains a Gaussian average of an odd number of fields, hence it is zero.

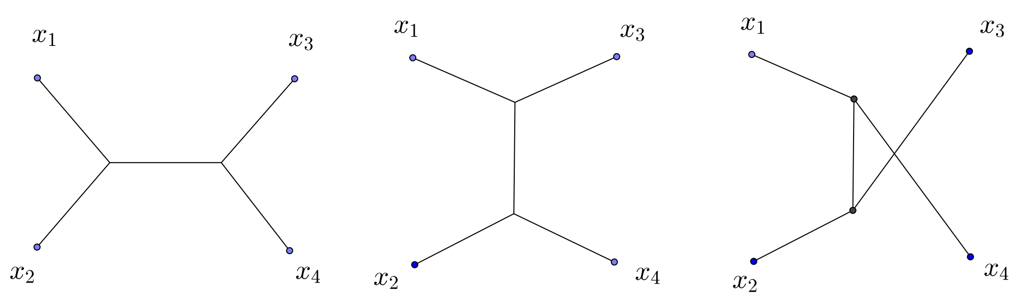
$O(g^2)$ contains the first nonzero connected contribution. It is given by

$$
\begin{aligned}
G_c^{(4)}(x_1,\ldots,x_4) \;&=\; \frac{1}{2}\left(\frac{-ig}{3!}\right)^2 \int d^4w_1 \int d^4w_2 \,\langle \phi^3(w_1)\phi^3(w_2)\phi(x_1)\ldots\phi(x_2)\rangle_c \\
&= \frac{1}{2}\left(\frac{-ig}{3!}\right)^2 \times 3!^2 \times 2 \int d^4w_1 \int d^4w_2 \,\{iD(x_1-w_1)iD(x_2-w_1)iD(w_1-w_2)iD(x_3-w_2)iD(x_4-w_2) \\
&\qquad + iD(x_1-w_1)iD(x_3-w_1)iD(w_1-w_2)iD(x_2-w_2)iD(x_4-w_2) \\
&\qquad + iD(x_1-w_1)iD(x_4-w_1)iD(w_1-w_2)iD(x_2-w_2)iD(x_3-w_2)\} \\
&= -ig^2 \int d^4w_1 \int d^4w_2 \,\{D(x_1-w_1)D(x_2-w_1)D(w_1-w_2)D(x_3-w_2)D(x_4-w_2) \\
&\qquad + D(x_1-w_1)D(x_3-w_1)D(w_1-w_2)D(x_2-w_2)D(x_4-w_2) \\
&\qquad + D(x_1-w_1)D(x_4-w_1)D(w_1-w_2)D(x_2-w_2)D(x_3-w_2)\}
\end{aligned}
\tag{7}
$$

The factor $3!^2 \times 2$ in the second line is due to the total number of equivalent Wick contractions of $x_{1,2,3,4}$ with $w_{1,2}$; $3!$ ways of contracting $x_{1,2}$ with w_1, $3!$ ways of contracting $x_{3,4}$ with w_2, and a factor of 2 due to the exchange of w_1 and w_2 in the integral.

b) (0.5 points) Draw for each term the corresponding Feynman diagram in coordinate space.

Solution:



3. (2 points) Consider the lagrangian density

$$
\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}m^2\phi^a\phi^a + \frac{1}{2\lambda}\sigma^2 - \frac{1}{2}\sigma\phi^a\phi^a\,, \tag{8}
$$

where ϕ^a, $a = 1, 2, \ldots, N$ are N real scalar fields and σ another scalar field (called auxiliary field). Show that if we eliminate σ by using its equation of motion, we end up with the lagrangian density

$$
\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}m^2\phi^a\phi^a - \frac{\lambda}{8}(\phi^a\phi^a)^2\,. \tag{9}
$$

Solution: Since the field σ has no derivative term the equation of motion reduces to a constraint equation

$$\frac{\delta \mathcal{L}}{\delta \sigma} = 0 \tag{10}$$

which gives

$$\sigma = \frac{\lambda}{2}\phi^a \phi^a \tag{11}$$

Replacing σ in $\mathcal{L}$, one obtains a lagrangian that only depends on the fields ϕ^a

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}m^2 \phi^a \phi^a + \frac{1}{2\lambda}\left(\frac{\lambda}{2}\phi^a\phi^a\right)^2 - \frac{1}{2}\frac{\lambda}{2}\phi^a\phi^a\phi^b\phi^b \\
&= \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}m^2 \phi^a \phi^a - \frac{\lambda}{8}(\phi^a\phi^a)^2 \,, \qquad (12)
\end{aligned}$$

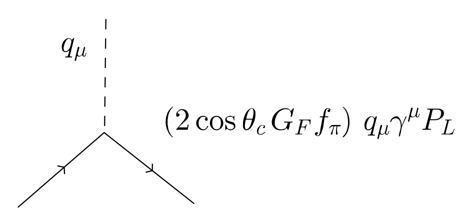
in agreement with the desired result.

4. (3 points total) The charged pion π^- can decay into a muon μ^- and a muon antineutrino $\bar{\nu}_\mu$ via the interaction

$$\mathcal{L}_I = 2\,\cos\theta_c\, G_F\, f_\pi\, \partial_\mu \phi\, \bar{\psi}^{(\mu)} \gamma^\mu P_L \psi^{(\nu)} \; + \text{ h.c.}\,,$$

where the complex scalar field ϕ represents the pion, the Dirac field $\psi^{(\mu)}$ represents the muon, the Dirac field $\psi^{(\nu)}$ represents the neutrino, $\cos\theta_c$ is the cosine of the *Cabibbo angle*, G_F is the *Fermi constant* and f_π is the *pion decay constant*. The chirality-projector $P_L = (1-\gamma_5)/2$ extracts the left-handed component of the neutrino field, where the neutrino is assumed to be massless.

a) (2.5 points) Compute the decay rate Γ for the process $\pi^-(q) \to \mu^-(p_1)\bar{\nu}_\mu(p_2)$ [**Show your work**], using the Feynman rule for the vertex



Hints: Using the energy projectors for spin 1/2 Dirac fermions

$$\begin{aligned}
\sum_{r=1,2} u_r(\vec{p})\bar{u}_r(\vec{p}) &= \not{p} + m \\
\sum_{r=1,2} v_r(\vec{p})\bar{v}_r(\vec{p}) &= \not{p} - m
\end{aligned}$$

the formula for the decay rate can be written as

$$\Gamma = \frac{1}{16\pi}\frac{m_\pi^2 - m_\mu^2}{m_\pi^3}\, X\,,$$

with m_π the pion mass, m_μ the muon mass, $m_\nu = 0$, and $X = \sum_{spin} \mathcal{A}^\dagger \mathcal{A}$ the squared amplitude summed over all final spin polarizations. You should obtain

$$\Gamma = \frac{G_F^2}{4\pi} f_\pi^2 \cos^2\theta_c m_\mu^2 m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

directly proportional to the squared muon mass.

Solution: For convenience we define $C \equiv 2\cos\theta_c G_F f_\pi$, so that the Feynman rule is $C \not{q} P_L$ with the pion momentum $q = p_1 + p_2$. The amplitude for the decay process is then

$$\begin{aligned}
\mathcal{A} &= C \bar{u}_r^{(\mu)}(p_1) \not{q} P_L v_s^{(\nu)}(p_2) \\
&= C \bar{u}_r^{(\mu)}(p_1) \left(\not{p}_1 P_L + P_R \not{p}_2\right) v_s^{(\nu)}(p_2)
\end{aligned} \tag{13}$$

where we used that $\not{p}_2 P_L = P_R \not{p}_2$, given that $\{\gamma_5, \gamma^\mu\} = 0$. We can now conveniently simplify the amplitude by using the Dirac equation for $\bar{u}_r^{(\mu)}(p_1)$ and $v_s^{(\nu)}(p_2)$ (they are external states for which the EoM holds true)

$$\begin{aligned}
&\bar{u}_r^{(\mu)}(p_1)(\not{p}_1 - m_\mu) = 0 \\
&\not{p}_2 v_s^{(\nu)}(p_2) = 0 \quad \textit{for} \quad m_\nu = 0\,.
\end{aligned} \tag{14}$$

Substitution in (13) leads to

$$\mathcal{A} = C\, m_\mu \bar{u}_r^{(\mu)}(p_1) P_L v_s^{(\nu)}(p_2)\,. \tag{15}$$

The hermitian conjugate amplitude is

$$\begin{aligned}
\mathcal{A}^\dagger &= C\, m_\mu \left(\bar{u}_r^{(\mu)}(p_1) P_L v_s^{(\nu)}(p_2)\right)^\dagger \\
&= C\, m_\mu \bar{v}_s^{(\nu)}(p_2) P_R u_r^{(\mu)}(p_1)
\end{aligned} \tag{16}$$

where we used $P_L^\dagger = P_L (\gamma_5^\dagger = \gamma_5)$ and $\gamma^0 P_L \gamma^0 = P_R$. The squared amplitude summed over the final spin polarisations is given by

$$\begin{aligned}
X &= \sum_{r,s=1,2} \mathcal{A}^\dagger \mathcal{A} = \sum_{r,s=1,2} C^2 m_\mu^2 \left(\bar{v}_s^{(\nu)}(p_2) P_R u_r^{(\mu)}(p_1)\right)\left(\bar{u}_r^{(\mu)}(p_1) P_L v_s^{(\nu)}(p_2)\right) \\
&= C^2 m_\mu^2 \mathrm{Tr}\left((\not{p}_1 - m_\mu) P_L \not{p}_2 P_R\right) = C^2 m_\mu^2 \mathrm{Tr}\left((\not{p}_1 - m_\mu)\not{p}_2 P_R\right) \\
&= \frac{1}{2} C^2 m_\mu^2 \mathrm{Tr}\left(\not{p}_1 \not{p}_2\right) = 2C^2 m_\mu^2 (p_1 p_2)
\end{aligned} \tag{17}$$

Squaring $q = p_1 + p_2$, with $q^2 = m_\pi^2$, $p_1^2 = m_\mu^2$ and $p_2^2 = 0$ one obtains

$$p_1 p_2 = \frac{1}{2}\left(q^2 - p_1^2 - p_2^2\right) = \frac{1}{2}\left(m_\pi^2 - m_\mu^2\right)\,. \tag{18}$$

The final squared amplitude is then

$$X = C^2 m_\mu^2 \left(m_\pi^2 - m_\mu^2\right) \tag{19}$$

proportional to the muon mass squared, as expected. Inserting X into the expression for Γ we obtain

$$\Gamma = \frac{G_F^2}{4\pi} f_\pi^2 \cos^2\theta_c m_\mu^2 m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 \tag{20}$$

in agreement with the provided result.

NB. The derivation of X can be equivalently carried out without the simplifications due to the use of the Dirac equation for the external spinors. The trace in X will then be slightly more complicated, but will obviously lead to the same result.

b) (0.5 points) Using $m_\pi = 139$ MeV, $m_\mu = 105.7$ MeV, $m_\nu = 0$, $G_F = 1.166 \times 10^{-5}$ GeV^{-2}, $\cos\theta_c = 0.974$, and knowing that the measured value of the charged pion lifetime $\tau = 1/\Gamma$ is 2.603×10^{-8} s, determine the value of f_π in MeV. [**Show your work**]

Solution: Using $\tau = 1/\Gamma = 2.603 \times 10^{-8}$ s and the conversion factor from seconds to GeV one obtains

$$\Gamma = 3.842 \times 10^7\,\mathrm{s}^{-1} = 2.528 \times 10^{-17} GeV = 2.528 \times 10^{-14} MeV \tag{21}$$

The pion decays constant is given by

$$f_\pi = \sqrt{\frac{4\pi\Gamma}{G_F^2 (\cos\theta_c)^2 m_\mu^2 m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2}} \tag{22}$$

Using $m_\pi = 139$ MeV, $m_\mu = 105.7$ MeV, $G_F = 1.166 \times 10^{-11}$ MeV^{-2}, $\cos\theta_c = 0.974$, one obtains f_π in MeV, $f_\pi = 94.43$ MeV (rounded to four significant digits, experimental uncertainties are not given).